

# CONDITIONING OF QUADRATIC HARNESSES

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**ABSTRACT.** We describe quadratic harnesses that arise through the double sided conditioning (bridges) of an already known quadratic harness and we characterize quadratic harnesses that arise by this construction from Lévy processes. We also analyze a construction that produces quadratic harnesses by "gluing together" two conditionally-independent  $q$ -Meixner processes. Our main tool is a deterministic time and space transformation which we use to classify a class of harnesses with quadratic conditional moments.

## 1. INTRODUCTION AND MAIN RESULTS

**1.1. Quadratic harness property.** Throughout the paper  $\mathcal{F} = (\mathcal{F}_{s,t})$  is a family of sigma fields with  $s < t$  from a nonempty open (generalized) interval  $\mathcal{T} = (T_0, T_1) \subset (-\infty, \infty)$  such that  $\mathcal{F}_{s,t} \subset \mathcal{F}_{r,u}$  for  $r, s, t, u \in \mathcal{T}$  with  $r \leq s \leq t \leq u$ . We include  $T_0 = -\infty$  or  $T_1 = \infty$  among the possible choices for the end-points of  $\mathcal{T}$ .

An integrable stochastic process  $\mathbf{X} = \{X_t : t \in \mathcal{T}\}$  is called a harness [14, 18, 20] on  $\mathcal{T}$  with respect to  $\mathcal{F}$  if  $X_u$  is  $\mathcal{F}_{s,t}$  measurable if  $u \geq t$  or if  $u \leq s$ , and for any  $s, t, u \in \mathcal{T}$  with  $s < t < u$ ,

$$(1.1) \quad \mathbb{E}(X_t | \mathcal{F}_{s,u}) = \frac{u-t}{u-s} X_s + \frac{t-s}{u-s} X_u.$$

All integrable Lévy processes are harnesses with respect to their natural filtration ([15, (2.8)]); additional examples appear in references on quadratic harnesses that are mentioned after Definition 1.1.

For a square-integrable process, a natural second-order extension of (1.1) is the requirement that  $\mathbb{V}\text{ar}(X_t | \mathcal{F}_{s,u})$  is a quadratic function of  $X_s, X_u$ . It turns out that this assumption is much more restrictive than (1.1) alone. Wesołowski [19] determined that there are only five such Lévy processes. Under certain additional assumptions [3, Theorem 2.2] asserts that there exist numerical constants  $\eta, \theta \in \mathbb{R}$   $\sigma, \tau \geq 0$  and  $\gamma \leq 1 + 2\sqrt{\sigma\tau}$  such that for all  $s < t < u$ ,

$$(1.2) \quad \mathbb{V}\text{ar}[X_t | \mathcal{F}_{s,u}] = F_{t,s,u} \left( 1 + \eta \frac{uX_s - sX_u}{u-s} + \theta \frac{X_u - X_s}{u-s} \right. \\ \left. + \sigma \frac{(uX_s - sX_u)^2}{(u-s)^2} + \tau \frac{(X_u - X_s)^2}{(u-s)^2} - (1-\gamma) \frac{(X_u - X_s)(uX_s - sX_u)}{(u-s)^2} \right),$$

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where

$$(1.3) \quad F_{t,s,u} = \frac{(u-t)(t-s)}{u(1+s\sigma) + \tau - s\gamma}.$$

In this paper, we adopt these formulas as the definition, but following [1, Corollary 5.1] for finite intervals  $T$  we allow negative values of parameters  $\sigma, \tau$ , and we allow any  $\gamma \geq -1$ .

**Definition 1.1.** Let  $\mathcal{T} = (T_0, T_1) \subset (0, \infty)$ . We will say that a square-integrable stochastic process  $\mathbf{X} = (X_t)_{t \in \mathcal{T}}$  is a quadratic harness on  $\mathcal{T}$  with respect to  $(\mathcal{F}_{r,s})$ , if  $X_u$  is  $\mathcal{F}_{s,t}$  measurable if  $u \geq t$  or if  $u \leq s$ ,

$$(1.4) \quad \mathbb{E}(X_t) = 0, \quad \mathbb{E}(X_t X_s) = \min\{t, s\}.$$

and equations (1.1), and (1.2) hold. When we want to indicate the parameters, we shall write  $\mathbf{X} \in QH(\eta, \theta; \sigma, \tau; \gamma)$ . (Formula (1.3) then follows by taking the expected value of (1.2), see Proposition 3.2.)

Examples of quadratic harnesses on  $(0, \infty)$  are five Lévy processes with quadratic conditional variances from [19]. Other examples include the classical versions of certain free Lévy processes ([5, Theorem 4.3]), classical versions of  $q$ -Brownian motion ([5, Theorem 4.1]), bi-Poisson process [4, 6, 7] and Markov processes with Askey-Wilson laws ([1, Theorem 1.1]).

*Remark 1.1 (Caution).* While the use of general time interval  $(T_0, T_1)$  is convenient for the purpose of studying transformations, the reader should be aware that most of the fundamental results (uniqueness, integrability, orthogonal polynomial martingales) valid for  $(0, \infty)$  do not extend automatically to quadratic harnesses on finite intervals, and may require additional assumptions on the interval and on the process. For example, from the proof of [3, Theorem 4.1] one can deduce that the moments of a quadratic harness on  $(0, T)$  with  $-1 < \gamma \leq 1 - 2\sqrt{\sigma\tau}$  are determined uniquely if the conditional moments of  $X_t$  and  $X_t^2$  with respect to past  $\sigma$ -field  $\mathcal{F}_s$  for  $s < t$  are linear and quadratic functions of  $X_s$  respectively. This result fails for intervals bounded away from 0, see Example 1.2 or when the assumption on one sided conditioning is dropped, see Example 1.1.

**1.2. Results.** Our main result shows how to transform a generic harness with quadratic conditional variances and with a product covariance into a quadratic harness.

**Theorem 1.1.** *Let  $\mathbf{X}$  be a harness (1.1) with respect to the family  $\mathcal{F}$  on an interval  $(T_0, T_1) \subset \mathbb{R}$  with mean*

$$E(X_t) = \alpha + \beta t$$

*and with covariance*

$$(1.5) \quad \text{Cov}(X_s, X_t) = (as + b)(ct + d), \quad s < t,$$

*such that  $ad - bc > 0$  and  $(at + b)(ct + d) > 0$  on  $(T_0, T_1)$ . Suppose that*

$$(1.6) \quad \text{Var}[X_t | \mathcal{F}_{s,u}] = F_{t,s,u} \left( \chi + \eta \frac{uX_s - sX_u}{u-s} + \theta \frac{X_u - X_s}{u-s} + \sigma \frac{(uX_s - sX_u)^2}{(u-s)^2} + \tau \frac{(X_u - X_s)^2}{(u-s)^2} + \rho \frac{(X_u - X_s)(uX_s - sX_u)}{(u-s)^2} \right),$$

where  $F_{t,s,u}$  is non-random, and

$$(1.7) \quad \tilde{\chi} := \chi + \alpha\eta + \theta\beta + \sigma\alpha^2 + \tau\beta^2 + \rho\alpha\beta > 0.$$

Let  $\psi(t) = (dt - b)/(a - ct)$ . Then stochastic process

$$(1.8) \quad Y_t = \frac{a - ct}{ad - bc} (X_{\psi(t)} - \alpha - \beta\psi(t))$$

is a quadratic harness in  $QH(\eta', \theta'; \sigma', \tau'; \gamma')$  on the interval  $\left(\frac{aT_0+b}{cT_0+d}, \frac{aT_1+b}{cT_1+d}\right) \subset (0, \infty)$ , and has parameters

$$(1.9) \quad \eta' = (d(\eta + \beta\rho + 2\alpha\sigma) + c(\theta + \alpha\rho + 2\beta\tau))/\tilde{\chi}$$

$$(1.10) \quad \theta' = (b(\eta + \beta\rho + 2\alpha\sigma) + a(\theta + \alpha\rho + 2\beta\tau))/\tilde{\chi}$$

$$(1.11) \quad \sigma' = (\tau c^2 + d\rho c + d^2\sigma)/\tilde{\chi}$$

$$(1.12) \quad \tau' = (\tau a^2 + b\rho a + b^2\sigma)/\tilde{\chi}$$

$$(1.13) \quad \gamma' = 1 + \frac{bc\rho + ad\rho + 2bd\sigma + 2ac\tau}{\tilde{\chi}}$$

Theorem 1.1 was motivated by a re-parametrizations of orthogonality measures of the related families of orthogonal polynomials when  $\sigma = 0$ , which was discovered by R. Szwarc; this observation is presented in an unpublished manuscript [2]. A version of Theorem 1.1 is implicit in the construction of quadratic harnesses from Markov processes based on the Askey-Wilson integral, see [1, Formula (2.28) and Section 6.2].

For reference, we state an elementary special case: if  $\mathbf{X} \in QH(\eta, \theta; \sigma, \tau; \gamma)$  and  $Z_t = aX_{t/a^2}$  with  $a \neq 0$ , then

$$(1.14) \quad \mathbf{Z} \in QH(\eta/a, a\theta; \sigma/a^2, a^2\tau; \gamma).$$

A related transformation  $Y_t = tX_{1/t}$  produces  $\mathbf{Y} \in QH(\theta, \eta; \tau, \sigma; \gamma)$ , i.e. entries within pairs  $(\eta, \theta)$  and  $(\sigma, \tau)$  are swapped. In fact, these two elementary facts can be used to re-write formulas (1.9–1.13) into other “equivalent forms”.

We also remark that the transformation used in Theorem 1.1 is reversible, so  $\mathbf{X}$  can be “represented” in terms of the quadratic harness  $\mathbf{Y} \in QH(\eta', \theta'; \sigma', \tau'; \gamma')$  as

$$X_t = (ct + d)Y_{(at+b)/(ct+d)} + \alpha + \beta t.$$

Next, we apply Theorem 1.1 to analyze/classify processes that arise through double-sided conditioning. For a (random or deterministic) real function  $\mathbf{X}$  of real parameter  $t$ , denote

$$\underline{\Delta}_{s,u}(\mathbf{X}) = \begin{bmatrix} \Delta_{s,u}(\mathbf{X}) \\ \tilde{\Delta}_{s,u}(\mathbf{X}) \end{bmatrix}$$

with

$$(1.15) \quad \Delta_{s,u}(\mathbf{X}) = \frac{X_u - X_s}{u - s} \text{ and } \tilde{\Delta}_{s,u}(\mathbf{X}) := \frac{uX_s - sX_u}{u - s}.$$

In the sequel, if  $\mathbf{X}$  is clear from the context, to shorten the notation we will sometimes write  $\underline{\Delta}_{s,u}$  instead of  $\underline{\Delta}_{s,u}(\mathbf{X})$ .

The right hand side of (1.2) is a quadratic polynomial in two real variables which we will write as

$$(1.16) \quad K(\underline{\Delta}_{s,u}) = \left(1 + \eta\tilde{\Delta}_{s,u} + \theta\Delta_{s,u} + \sigma\tilde{\Delta}_{s,u}^2 + \tau\Delta_{s,u}^2 - (1 - \gamma)\Delta_{s,u}\tilde{\Delta}_{s,u}\right).$$

It will be convenient to write  $K(a, b)$  instead of  $K\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$ .

**Theorem 1.2.** *Suppose  $(X_t)_{t \in (T_0, T_1)} \in QH(\eta, \theta; \sigma, \tau; \gamma)$ . Choose  $R < V$  in  $(T_0, T_1)$ . Then  $V(1 + R\sigma) + \tau - R\gamma > 0$  and*

$$M = M(X_R, X_V) = \sqrt{K(\underline{\Delta}_{R,V}(\mathbf{X}))} / \sqrt{V(R\sigma + 1) + \tau - R\gamma}$$

is non-zero with probability one. Define

$$(1.17) \quad Y_t = \frac{\sqrt{V}}{(V - R)M} ((1 + t/V)X_{(t+R)/(1+t/V)} - tX_V/V - X_R)$$

for  $t > 0$ . Then for  $0 < s < t < \infty$ , conditional version of (1.4) holds:

$$E(Y_t | \mathcal{F}_{R,V}) = 0; E(Y_s Y_t | \mathcal{F}_{R,V}) = s.$$

With  $\mathcal{G}_{s,u} = \mathcal{F}_{(s+R)/(1+s/V), (u+R)/(1+u/V)}$ , condition (1.1) holds: for  $s < t < u$ ,

$$E(Y_t | \mathcal{G}_{s,u}) = \frac{u-t}{u-s} Y_s + \frac{t-s}{u-s} Y_u$$

and condition (1.2) holds: for  $s < t < u$ ,

$$(1.18) \quad \text{Var}[Y_t | \mathcal{G}_{s,u}] = F_{t,s,u} \left( 1 + \tilde{\eta} \frac{uY_s - sY_u}{u-s} + \tilde{\theta} \frac{Y_u - Y_s}{u-s} + \tilde{\sigma} \frac{(uY_s - sY_u)^2}{(u-s)^2} + \tilde{\tau} \frac{(Y_u - Y_s)^2}{(u-s)^2} - (1 - \tilde{\gamma}) \frac{(Y_u - Y_s)(uY_s - sY_u)}{(u-s)^2} \right),$$

with  $\mathcal{F}_{R,V}$ -measurable parameters:

$$(1.19) \quad \tilde{\eta} = \frac{-\theta + V\eta - 2\tau\Delta_{R,V}(\mathbf{X}) + 2\sigma V\tilde{\Delta}_{R,V}(\mathbf{X}) - (1 - \gamma)(V\Delta_{R,V}(\mathbf{X}) - \tilde{\Delta}_{R,V}(\mathbf{X}))}{\sqrt{V}M(V(R\sigma + 1) + \tau - R\gamma)},$$

$$(1.20) \quad \tilde{\theta} = \frac{\sqrt{V}(\theta - R\eta + 2\tau\Delta_{R,V}(\mathbf{X}) - 2R\sigma\tilde{\Delta}_{R,V}(\mathbf{X}) - (1 - \gamma)(\tilde{\Delta}_{R,V}(\mathbf{X}) - R\Delta_{R,V}(\mathbf{X})))}{M(V(R\sigma + 1) + \tau - R\gamma)},$$

and with non-random

$$(1.21) \quad \tilde{\sigma} = \frac{\sigma V^2 + (1 - \gamma)V + \tau}{V(V(R\sigma + 1) + \tau - R\gamma)},$$

$$(1.22) \quad \tilde{\tau} = V \frac{\sigma R^2 + (1 - \gamma)R + \tau}{V(R\sigma + 1) + \tau - R\gamma},$$

$$(1.23) \quad \tilde{\gamma} = \frac{V\gamma - R(V\sigma + 1) - \tau}{V(R\sigma + 1) + \tau - R\gamma} = -1 + \frac{(V - R)(\gamma + 1)}{R\sigma V + V - R\gamma + \tau}.$$

(Constant  $F_{t,s,u}$  is then given by formula (1.3) with  $\tilde{\sigma}, \tilde{\tau}, \tilde{\gamma}$  replacing  $\sigma, \tau, \gamma$ .)

We now specialize this same result to Markov processes. Notice that quadratic harnesses have cadlag versions, so regular versions of conditional distributions exist. If  $\mathbf{Z}$  is a quadratic harness with respect to  $\{\mathcal{F}_{s,t}\}$  on  $(T_0, T_1)$  and  $R < V$  are in  $(T_0, T_1)$ , then conditionally on  $Z(V), Z(R)$ , the process still has quadratic conditional variances with respect to  $\mathcal{F}_{s,u}$  when  $s, u \in (V, R)$ , and of course the

conditioned process takes deterministic values at the endpoints  $V$  and  $R$ . Furthermore, the conditional variance (1.2) is positive with probability one. Thus there is a set of probability one of pairs  $z_R \in \text{supp}(Z(R))$ ,  $z_V \in \text{supp}(Z(V))$  such that the laws

$$(1.24) \quad \pi_t(U) := \Pr(Z(t) \in U | Z(R) = z_R, Z(V) = z_V)$$

are well defined for all  $t \in (V, R)$  and that there are Borel sets  $U_s$  of  $\pi_s$ -measure one such that

$$(1.25) \quad P_{s,t}(x, U) := \Pr(Z(t) \in U | Z(s) = x, Z(V) = z_V)$$

are well defined for all  $s < t$  in  $(V, R)$  and all  $x \in U_s$ . It is known that (1.24) and (1.25) determine a Markov process if  $\mathbf{Z}$  is Markov, or more generally, if  $\mathbf{Z}$  is the so called reciprocal process, see [17, Theorem 4.1].

The following is the corresponding re-statement of Theorem 1.2.

**Theorem 1.2'.** *Let  $\mathbf{Z} = (Z_t)_{t \in (T_0, T_1)} \in QH(\eta, \theta; \sigma, \tau; \gamma)$  be Markov. Fix  $R < V$  in  $(T_0, T_1)$  and  $z_R \in \text{supp}(Z_R)$ ,  $z_V \in \text{supp}(Z_V)$  such that with  $\Delta_{RV} := (z_V - z_R)/(V - R)$  and  $\tilde{\Delta}_{RV} := (Vz_R - Rz_V)/(V - R)$ , we have  $K(\Delta_{RV}, \tilde{\Delta}_{RV}) > 0$  and such that (1.24) and (1.25) are well defined. Let  $\mathbf{X} = \{X(t) : R \leq t \leq V\}$  be the conditional process, i.e. the Markov process with univariate laws (1.24) and transition probabilities (1.25).*

*Then  $V(1 + R\sigma) + \tau - R\gamma > 0$  and*

$$M^2 = K(\Delta_{RV}, \tilde{\Delta}_{RV})/(V(1 + R\sigma) + \tau - R\gamma) > 0.$$

*Process  $\mathbf{X}$  is related by a deterministic transformation*

$$(1.26) \quad Y_t = \frac{\sqrt{V}}{(V - R)M} \left( (t/V + 1)X_{(t+R)/(t/V+1)} - (t \frac{z_V}{V} + z_R) \right)$$

*(compare (1.17)) to the quadratic harness  $\mathbf{Y} \in QH(\tilde{\eta}, \tilde{\theta}; \tilde{\sigma}, \tilde{\tau}; \tilde{\gamma})$  on  $(0, \infty)$  with parameters given by (1.20–1.23) with  $\underline{\Delta}_{RV}(\mathbf{X}) = [\Delta_{RV}, \tilde{\Delta}_{RV}]^T$  which is now a non-random vector.*

*Remark 1.2.* Analogous results with essentially the same proof hold for one sided conditioning on  $(0, V)$  and on  $(R, \infty)$ . Formulas for parameters in these two cases correspond to Theorem 1.2 after taking the limit as  $R \rightarrow 0$  or as  $V \rightarrow \infty$ , respectively, with conventions that  $\lim_{V \rightarrow \infty} \Delta_{RV} = 0$ ,  $\lim_{R \rightarrow 0} \Delta_{RV} = z_V/V$ ,  $\lim_{V \rightarrow \infty} \tilde{\Delta}_{RV} = z_R$ ,  $\lim_{R \rightarrow 0} \tilde{\Delta}_{RV} = 0$ . More specifically, suppose  $\mathbf{Z}$  is a quadratic harness on  $(0, \infty)$  in  $QH(\eta, \theta; \sigma, \tau; \gamma)$ .

- (i) Let  $\mathbf{X} = \{X(t) : 0 \leq t \leq V\}$  be the conditional process arising by conditioning  $\mathbf{Z}$  with respect to  $Z_V = z_V$ . Assume that

$$\kappa^2 = \left(1 + \frac{\tau}{V}\right) \left(1 + \theta \frac{z_V}{V} + \tau \frac{z_V^2}{V^2}\right) > 0.$$

Then

$$(1.27) \quad Y_t = \frac{1 + \tau/V}{\kappa} \left( (1 + t/V)X_{t/(1+t/V)} - \frac{t}{V}z_V \right)$$

is a quadratic harness with parameters

$$(1.28) \quad \theta_Y = \frac{\theta + 2\tau z_V/V}{\kappa},$$

$$(1.29) \quad \eta_Y = \frac{-\theta + V\eta - 2\tau z_V/V - (1 - \gamma)z_V}{V\kappa},$$

$$(1.30) \quad \tau_Y = \frac{V\tau}{V + \tau},$$

$$(1.31) \quad \sigma_Y = \frac{\sigma V^2 + (1 - \gamma)V + \tau}{V(V + \tau)},$$

$$(1.32) \quad \gamma_Y = \frac{V\gamma - \tau}{V + \tau}.$$

- (ii) Let  $\mathbf{X} = \{X(t) : R \leq t < \infty\}$  be the conditional process arising by conditioning  $\mathbf{Z}$  with respect to  $Z_R = z_R$ . Assume that

$$\kappa^2 = (1 + R\sigma)(1 + \eta z_R + \sigma z_R^2) > 0.$$

For  $0 < t < \infty$ , let

$$Y_t = (1 + R\sigma)(X_{t+R} - z_R)/\kappa.$$

Then  $\mathbf{Y}$  is a quadratic harness with parameters

$$(1.33) \quad \theta_Y = \frac{\theta - R\eta - 2R\sigma z_R - (1 - \gamma)z_R}{\kappa},$$

$$(1.34) \quad \eta_Y = \frac{\eta + 2\sigma z_R}{\kappa},$$

$$(1.35) \quad \tau_Y = \frac{\sigma R^2 + (1 - \gamma)R + \tau}{1 + R\sigma},$$

$$(1.36) \quad \sigma_Y = \frac{\sigma}{1 + R\sigma},$$

$$(1.37) \quad \gamma_Y = \frac{\gamma - R\sigma}{1 + R\sigma}.$$

Conditioning allows us to identify new quadratic harnesses, as seen from the following.

**Theorem 1.3.** *Suppose that  $\sigma, \tau > 0$  are such that  $\sigma\tau < 1$ ,  $\gamma = 1 - 2\sqrt{\sigma\tau}$  and  $\eta, \theta$  are real numbers such that  $\sqrt{\tau}\eta + \sqrt{\sigma}\theta = 0$ . Then there exists a square-integrable Markov process  $(X_t)_{t \in (0, \infty)}$  such that (1.1), (1.2), and (1.4) hold.*

The range of parameters in Theorem 1.3 corresponds formally to  $q = 1$  in [1], and perhaps could be obtained from the processes in that paper by a limiting procedure.

Proofs of Theorems 1.1 and 1.2' are in Section 3.2. Theorem 1.3 is proved in Section 4.2.

**1.3. Examples.** Here we use Theorem 1.1 to give examples that illustrate lack of uniqueness for quadratic harnesses on finite intervals.

**Example 1.1.** Let  $(W_t)$  be the Wiener process and  $\xi$  be a centered random variable independent of  $W$  with  $\mathbb{E}\xi^2 = v^2$ . Let

$$X_t = W_t + \xi t, \quad t > 0.$$

Then  $E(X_t) = 0$  and  $\text{Cov}(X_s, X_t) = s(1 + v^2 t)$ . Furthermore,  $X_t$  is a harness with respect to its natural sigma fields, and

$$\mathbb{V}\text{ar}(X_t | \mathcal{F}_{s,t}) = \mathbb{V}\text{ar}(W_t | W_s, W_u) = F_{t,s,u}.$$

So from Theorem 1.1 (or by direct calculation) we see that

$$Y_t = (1 - tv^2)X_{1/(1-tv^2)}$$

is a quadratic harness on  $(0, 1/v^2)$  with respect to natural  $\sigma$ -fields and has parameters  $\eta = \theta = \sigma = \tau = 0$ ,  $\gamma = 1$ .

**Example 1.2.** This example is a time-inversion of Example 1.1. Let  $(W_t)$  be the Wiener process and  $\xi$  be a centered random variable independent of  $W$  with  $\mathbb{E}\xi^2 = v^2$ . Then, with respect to natural  $\sigma$ -fields,  $X_t = W_{t-v^2} + \xi$  is a quadratic harness on  $(v^2, \infty)$  with parameters  $\eta = \theta = \sigma = \tau = 0$ ,  $\gamma = 1$ . It is also easy to see that  $X_t$  and  $X_t^2 - t$  are martingales with respect to the past  $\sigma$ -field.

Next, we give a simple example of a quadratic harnesses with  $\gamma > 1$  and  $\sigma\tau > 1$ ; such examples are interesting because most of the general theory developed in [3] does not apply.

**Example 1.3.** Suppose  $(G_t)_{t \geq 0}$  is a gamma process with parameters  $(1, 1)$  (see Example 4.1 below). Let  $\xi$  be an independent random variable with mean  $E(\xi) = \beta > 0$  and  $E\xi^2 = v^2$ . Let

$$X_t = \xi G_t$$

Then  $\mathbb{E}(X_t) = \beta t$ . From

$$\text{Cov}(X_s, X_t) = \mathbb{E}(\text{Cov}(X_s, X_t | \xi)) + \text{Cov}(\mathbb{E}(X_s | \xi), \mathbb{E}(X_t | \xi))$$

we see that for  $s \leq t$ ,  $\text{Cov}(X_s, X_t) = s(v^2 t + \beta)$ . Let  $\mathcal{F}_{s,u}$  be the natural  $\sigma$ -fields associated with  $X_t$ . Consider the auxiliary  $\sigma$ -fields  $\tilde{\mathcal{F}}_{s,u}$  generated by  $\xi$  and  $\{G_t : t \in (0, s] \cup [u, \infty)\}$ . Then  $\mathbb{E}(X_t | \tilde{\mathcal{F}}_{s,u}) = \xi \left( \frac{u-t}{u-s} G_s + \frac{t-s}{u-s} G_u \right) = \frac{u-t}{u-s} X_s + \frac{t-s}{u-s} X_u$  so  $E(X_t | \mathcal{F}_{s,u}) = \frac{u-t}{u-s} X_s + \frac{t-s}{u-s} X_u$ . Similarly, using (4.4) with  $\alpha = 1$  we get

$$\begin{aligned} \mathbb{V}\text{ar}(X_t | \tilde{\mathcal{F}}_{s,u}) &= \xi^2 \mathbb{V}\text{ar}(G_t | G_s, G_u) = \xi^2 \frac{(u-t)(t-s)}{(u-s+1)(u-s)^2} (G_u - G_s)^2 \\ &= \frac{(u-t)(t-s)}{(u-s+1)(u-s)^2} (X_u - X_s)^2, \end{aligned}$$

so

$$\mathbb{V}\text{ar}(X_t | \mathcal{F}_{s,u}) = \frac{(u-t)(t-s)}{(u-s+1)(u-s)^2} (X_u - X_s)^2.$$

From Theorem 1.1 applied with

$$a = v, \quad b = 0, \quad c = v, \quad d = \beta/v$$

we see that

$$Z_t = v(1-t)X_{\frac{\beta t}{v^2(1-t)}} - \frac{\beta^2}{v}t$$

is a quadratic harness on  $(0, 1)$  with parameters

$$\eta = \theta = 2v/\beta, \quad \sigma = \tau = v^2/\beta^2, \quad \gamma = 1 + 2\sqrt{\sigma\tau}.$$

In particular,  $\gamma = 1 + 2\sqrt{\sigma\tau}$  and  $\sigma\tau = v^4/\beta^4 = (E(\xi^2))^2/(E(\xi))^4 \geq 1$  can be arbitrarily large.

Of course, the distribution of  $\xi$  is arbitrary so the moments of  $Z_t$  are not determined uniquely and may fail to exist.

An example of quadratic harness on  $(0, \infty)$  with  $\gamma = 1 + 2\sqrt{\sigma\tau}$  is a bi-Pascal process [16], which is a quadratic harness on  $(0, \infty)$  with arbitrary  $\sigma = \tau > 0$  and arbitrary  $\eta = \theta > 2\sqrt{\tau}$ . A related quadratic harness on a finite interval  $(0, T)$  arises as a transformation (1.8) of the generalized Waring process introduced in [9] and [8], see also [22] and [21].

## 2. MATRIX NOTATION

For calculations, it will be convenient to parametrize time as a subset of a projective plane, i.e. using  $\underline{t} = \begin{bmatrix} t \\ 1 \end{bmatrix}$ . We rewrite (1.1) in vector form as

$$(2.1) \quad \mathbb{E}(X_t | \mathcal{F}_{s,u}) = \langle \underline{t}, \underline{\Delta}_{s,u}(\mathbf{X}) \rangle ,$$

where  $\langle \underline{a}, \underline{b} \rangle = \underline{a}^T \underline{b}$ , and the components of  $\underline{\Delta}_{s,u}(\mathbf{X})$  are defined by (1.15).

It follows from (1.1) that admissible expectations of a harness  $\mathbf{X}$  are affine in  $t$ , i.e.,

$$(2.2) \quad \mathbb{E}(X_t) = \langle \underline{t}, \underline{\mu} \rangle , \quad t \in \mathcal{T} ,$$

where

$$(2.3) \quad \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} .$$

Moreover, if  $\mathbf{X}$  is a square integrable harness then by [3, Proposition 2.1] the admissible covariances are of the form

$$(2.4) \quad \mathbb{Cov}(X_s, X_t) = \langle \underline{s}, \Sigma \underline{t} \rangle , \quad s, t \in \mathcal{T}, \quad s \leq t ,$$

where

$$\Sigma = \begin{bmatrix} c_0 & c_1 \\ c_2 & c_3 \end{bmatrix} \quad \text{and} \quad \underline{s} = \begin{bmatrix} s \\ 1 \end{bmatrix} .$$

Throughout this paper, letters  $s, t, u \in \mathcal{T}$  are reserved to denote time, and  $\underline{s}, \underline{t}$ , and also  $\underline{u} = [u, 1]^T$  have this special meaning also when used with subscripts or primed. We also use the convention that  $s \leq t \leq u$ .

Note that under our convention  $s \leq t$  so  $\Sigma$  is not a symmetric matrix; for example, covariance  $s \wedge t$  is represented by matrix  $\Sigma = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

Formula (1.2) in matrix form can be written as

$$(2.5) \quad \mathbb{Var}(X_t | \mathcal{F}_{s,u}) = F_{t,s,u} \left( 1 + \langle \underline{\theta}, \underline{\Delta}_{s,u} \rangle + \langle \underline{\Delta}_{s,u}, \Gamma \underline{\Delta}_{s,u} \rangle \right) ,$$

where

$$(2.6) \quad \underline{\theta} = \begin{bmatrix} \theta \\ \eta \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \tau & -1 \\ \gamma & \sigma \end{bmatrix} .$$

Here  $\eta, \theta, \sigma, \tau, \gamma$  are constants independent of  $s, t, u$ .

Of course matrix  $\Gamma$  is determined only up to  $\Gamma_{1,2} + \Gamma_{2,1}$ . The usual choice of symmetric  $\Gamma$  is in fact inconvenient, see Proposition 3.2. The choice made in (2.6) matches the notation we used in previous papers: after substituting  $q$  for  $\gamma$ , the resulting parametrization of the conditional variance is identical to [3, (2.14)].

The non-random constant  $F_{t,s,u}$  is determined uniquely by taking the average of both sides of (2.5). According to [3, (2.15)], with the choice of  $\Gamma$  as in (2.6),



formula (1.3) holds. We re-write formula (1.3) in matrix notation using a special matrix

$$(2.7) \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

It is easy to see that  $J^2 = -I$ ,  $J^T = -J$ . For future reference we state also two less obvious properties: for  $A \in GL_2(\mathbb{R})$ ,

$$(2.8) \quad A^T J A = \det(A) J \quad \text{and} \quad J^T A J = \det(A) (A^{-1})^T.$$

Formula (1.3) can now be written as

$$(2.9) \quad F_{t,s,u} = \frac{\langle \underline{\mathbf{t}}, J \underline{\mathbf{u}} \rangle \langle \underline{\mathbf{s}}, J \underline{\mathbf{t}} \rangle}{\langle \underline{\mathbf{s}}, J^T \Gamma J \underline{\mathbf{u}} \rangle}.$$

This formula makes sense for any  $2 \times 2$  matrix  $\Gamma$  as long as the denominator is non-zero.

Finally, we note for future reference that the conditional covariance for quadratic harnesses also takes a simple form: with  $s < t_1 < t_2 < u$ ,

$$(2.10) \quad \text{Cov}(X_{t_1}, X_{t_2} | \mathcal{F}_{s,u}) = \frac{\langle \underline{\mathbf{t}}_2, J \underline{\mathbf{u}} \rangle \langle \underline{\mathbf{s}}, J \underline{\mathbf{t}}_1 \rangle}{\langle \underline{\mathbf{s}}, J^T \Gamma J \underline{\mathbf{u}} \rangle} K(\underline{\Delta}_{\mathbf{s}, \mathbf{u}}),$$

where  $K(\underline{\mathbf{a}}) = 1 + \langle \underline{\theta}, \underline{\mathbf{a}} \rangle + \langle \underline{\mathbf{a}}, \Gamma \underline{\mathbf{a}} \rangle$  is the quadratic polynomial from (1.16) and (2.5). A quick way to see this is to notice that (1.1) implies

$$\text{Cov}(X_{t_1}, X_{t_2} | \mathcal{F}_{s,u}) = \frac{u - t_2}{u - t_1} \text{Var}(X_{t_1} | \mathcal{F}_{s,u}) = \frac{\langle \underline{\mathbf{t}}_2, J \underline{\mathbf{u}} \rangle}{\langle \underline{\mathbf{t}}_1, J \underline{\mathbf{u}} \rangle} \text{Var}(X_{t_1} | \mathcal{F}_{s,u}).$$

### 3. DETERMINISTIC TIME AND SPACE TRANSFORMATION

With a non-degenerate affine function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , written in matrix notation as

$$(3.1) \quad f(x, y) = [x, y] A + [m_1, m_2],$$

we associate Möbius transform  $\varphi(t) = (at + b)/(ct + d)$  generated by

$$A = A_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R}).$$

If  $\mathbf{X}$  is a stochastic process on an open interval  $\mathcal{T} \subset \mathbb{R}$ , and  $\mathcal{T}$  lies in the range of  $\varphi$ , we define a deterministic transformation  $\mathbf{X}^f$  of the stochastic process  $\mathbf{X}$  as the process  $\mathbf{Y} = \mathbf{X}^f$  on  $\mathcal{S} = \varphi^{-1}(\mathcal{T})$  such that

$$(3.2) \quad Y_t = (ct + d)X_{\varphi(t)} + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}} \rangle, \quad t \in \mathcal{S}.$$

We note that  $\varphi$  is increasing on  $\mathcal{S}$  if  $\det(A) > 0$  and it is decreasing otherwise.

Our interest in this transformation comes from the fact that special cases appeared as ad hoc tricks in constructions of quadratic harnesses in [6, 1, 16].

A calculation verifies that as long as the time domains of the processes match,

$$(\mathbf{X}^f)^g = \mathbf{X}^{g \circ f}.$$

This allows us to build more complicated transformations in simple steps, and gives us flexibility to consider either  $\mathbf{Y} = \mathbf{X}^f$  or  $\mathbf{X} = \mathbf{Y}^{f^{-1}}$  as needed.

For non-degenerate affine  $f$  with Möbius transform  $\varphi : (T_0, T_1) \rightarrow (T'_0, T'_1)$  and  $s < t$  in  $\mathcal{T}$ , the transformed  $\sigma$ -fields are

$$(3.3) \quad \mathcal{F}_{s,u}^f = \begin{cases} \mathcal{F}_{\varphi(s), \varphi(u)} & \text{if } \det(A) > 0, \\ \mathcal{F}_{\varphi(u), \varphi(s)} & \text{if } \det(A) < 0. \end{cases}$$

It is clear that if  $\mathbf{X}$  has linear regressions and quadratic conditional variances with respect to  $\mathcal{F}_{s,u}$  then  $\mathbf{X}^f$  has linear regressions and quadratic conditional variances with respect to  $\mathcal{F}_{s,u}^f$ . The following technical result describes how the parameters transform in the setting slightly more general than Theorem 1.1.

**Proposition 3.1.** *Let  $\mathbf{X}$  be a harness (1.1) with respect to the family  $\mathcal{F}$  on an interval  $(T_0, T_1)$  with the first two moments given by (2.2) and (2.4). Suppose*

$$(3.4) \quad \mathbb{V}\text{ar}(X_t | \mathcal{F}_{s,u}) = F_{t,s,u} (\chi + \langle \underline{\theta}, \underline{\Delta}_{s,u} \rangle + \langle \underline{\Delta}_{s,u}, \Gamma \underline{\Delta}_{s,u} \rangle),$$

with non-random  $F_{t,s,u}$ ,  $\chi \in \mathbb{R}$ ,  $\underline{\theta} \in \mathbb{R}^2$ , and arbitrary  $2 \times 2$  matrix  $\Gamma$ . Let  $f$  be a non-degenerate affine function with matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , corresponding Möbius transform  $\varphi$  and shift  $\underline{\mathbf{m}}$ . If  $\varphi$  is well defined on the entire interval  $(T_0, T_1)$  then  $\tilde{\mathbf{X}} := \mathbf{X}^f$  given by

$$\tilde{X}_t = (ct + d)X_{(at+b)/(ct+d)} + m_1 t + m_2$$

also satisfies (2.2), (2.4) with

$$(3.5) \quad \tilde{\underline{\mu}} = A^T \underline{\mu} + \underline{\mathbf{m}}.$$

$$(3.6) \quad \tilde{\Sigma} = \begin{cases} A^T \Sigma A & \text{if } \det(A) > 0, \\ A^T \Sigma^T A & \text{if } \det(A) < 0. \end{cases}$$

and (1.1) and (3.4) with

$$(3.7) \quad \tilde{\Gamma} = \begin{cases} A^{-1} \Gamma (A^{-1})^T & \text{if } \det(A) > 0, \\ (A^{-1})^T \Gamma A^{-1} & \text{if } \det(A) < 0. \end{cases}$$

$$(3.8) \quad \tilde{\underline{\theta}} = A^{-1} \underline{\theta} - (\tilde{\Gamma} + \tilde{\Gamma}^T) \underline{\mathbf{m}}.$$

and

$$(3.9) \quad \tilde{\chi} = \chi - \langle \tilde{\underline{\theta}}, \underline{\mathbf{m}} \rangle - \langle \underline{\mathbf{m}}, \tilde{\Gamma} \underline{\mathbf{m}} \rangle,$$

We remark that in the most interesting case of "product covariance"

$$(3.10) \quad \text{Cov}(X_s, X_t) = (\varepsilon s + \delta)(\phi t + \psi), \quad s < t,$$

transformation (3.6) preserves this product form. In fact, in this case

$$\Sigma = \begin{bmatrix} \varepsilon \phi & \varepsilon \psi \\ \delta \phi & \delta \psi \end{bmatrix},$$

but if we collect the coefficients of the covariance (3.10) into another matrix

$$\Theta = \begin{bmatrix} \varepsilon & \delta \\ \phi & \psi \end{bmatrix},$$

then a calculation based on (3.6) shows that for  $\det A > 0$  the covariance of  $\tilde{\mathbf{X}}$  corresponds to  $\tilde{\Theta} = \Theta A$ .

We postpone the proof of Proposition 3.1 until Section 3.1, so that we can first clarify the role of non-random constant  $F_{t,s,u}$ . The main point is that in the non-degenerate case with  $c_1 > c_2$ , this constant is determined uniquely by taking the average of both sides of (3.4). Furthermore,  $F_{t,s,u}$  is often given by formula (2.9), i.e., (1.3).

**Proposition 3.2.** *Suppose a harness  $\mathbf{X}$  has mean (2.2) and non-degenerate covariance (2.4) with  $c_1 > c_2$ . If  $\mathbf{X}$  has quadratic conditional variance (3.4) and the off-diagonal entries of matrix  $\Gamma$  are chosen so that*

$$(3.11) \quad \chi + \langle \underline{\theta}, \underline{\mu} \rangle + \langle \underline{\mu}, \Gamma \underline{\mu} \rangle + \text{tr}(\Gamma \Sigma^T) = 0,$$

*then  $\langle \underline{s}, J^T \Gamma J \underline{u} \rangle = u(1 + s\sigma) + \tau - s\gamma \neq 0$  and  $F_{t,s,u}$  is given by formula (2.9). Moreover, transformation formulas in Proposition 3.1 preserve (2.9).*

Formulas (1.2) and (2.6) illustrate the choice of such  $\Gamma$ .

### 3.1. Proof of Propositions 3.1 and 3.2.

**Lemma 3.3.** *Let  $f$  be a non-degenerate affine function (3.1) with Möbius transform  $\varphi$ . If  $s' = \varphi(s)$ ,  $u' = \varphi(u)$ , then*

$$(3.12) \quad \underline{\Delta}_{s,u}(\mathbf{X}^f) = A^T \underline{\Delta}_{s',u'}(\mathbf{X}) + \underline{\mathbf{m}}.$$

*Proof.* Let  $g(x, y) = [x, y]A$  denote the linear part of  $f$ . Since  $\underline{\Delta}_{s,u}(a) = \underline{\mathbf{m}}$  on a linear function  $a(t) := \langle \underline{\mathbf{t}}, \underline{\mathbf{m}} \rangle$ , and  $\mathbf{X}^f(t) = \mathbf{X}^g(t) + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}} \rangle$ , we have  $\underline{\Delta}_{s,u}(\mathbf{X}^f) = \underline{\Delta}_{s,u}(\mathbf{X}^g) + \underline{\mathbf{m}}$ . Since  $X_{s'} = \mathbf{X}^g(s)/(cs + d)$ , and from the matrix form of (1.15) we have

$$(3.13) \quad \underline{\Delta}_{s,t}(\mathbf{X}) = \frac{J(X_t \underline{\mathbf{s}} - X_s \underline{\mathbf{t}})}{\langle \underline{\mathbf{t}}, J \underline{\mathbf{s}} \rangle},$$

we get

$$\underline{\Delta}_{s',u'}(\mathbf{X}) = J \frac{\frac{\mathbf{X}^g(u)}{cu+d} \underline{\mathbf{s}}' - \frac{\mathbf{X}^g(s)}{cs+d} \underline{\mathbf{u}}'}{\langle \underline{\mathbf{u}}', J \underline{\mathbf{s}}' \rangle} = J \frac{\mathbf{X}^g(u)(cs+d) \underline{\mathbf{s}}' - \mathbf{X}^g(s)(cu+d) \underline{\mathbf{u}}'}{\langle (cu+d) \underline{\mathbf{u}}', J(cs+d) \underline{\mathbf{s}}' \rangle}.$$

Noting that

$$(3.14) \quad (cs+d) \underline{\mathbf{s}}' = A \underline{\mathbf{s}},$$

and using (2.8) we get

$$\begin{aligned} \underline{\Delta}_{s',u'}(\mathbf{X}) &= JA \frac{\mathbf{X}^g(u) \underline{\mathbf{s}} - \mathbf{X}^g(s) \underline{\mathbf{u}}}{\langle \underline{\mathbf{u}}, A^T J A \underline{\mathbf{s}} \rangle} = (A^{-1})^T A^T J A \frac{\mathbf{X}^g(u) \underline{\mathbf{s}} - \mathbf{X}^g(s) \underline{\mathbf{u}}}{\langle \underline{\mathbf{u}}, A^T J A \underline{\mathbf{s}} \rangle} \\ &= (A^{-1})^T \underline{\Delta}_{s,u}(\mathbf{X}^g). \end{aligned}$$

Thus  $\underline{\Delta}_{s,u}(\mathbf{X}^f) = \underline{\Delta}_{s,u}(\mathbf{X}^g) + \underline{\mathbf{m}} = A^T \underline{\Delta}_{s',u'}(\mathbf{X}) + \underline{\mathbf{m}}$ .  $\square$

*Proof of Proposition 3.1.* Throughout the proof we write  $t' = \varphi(t)$ . If  $\varphi$  is increasing, by (2.1) and the definition of  $\mathbf{X}^f$  we have

$$(3.15) \quad \begin{aligned} \mathbb{E}(\mathbf{X}^f(t) | \mathcal{F}_{s,u}^f) &= (ct+d) \mathbb{E}(X_{t'} | \mathcal{F}_{s',u'}) + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}} \rangle \\ &= (ct+d) \langle \underline{\mathbf{t}}', \underline{\Delta}_{s',u'}(\mathbf{X}) \rangle + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}} \rangle. \end{aligned}$$

By (3.14) and (3.12) we get

$$\mathbb{E}(\mathbf{X}^f(t) | \mathcal{F}_{s,u}^f) = \langle A \underline{\mathbf{t}}, (A^{-1})^T (\underline{\Delta}_{s,u}(\mathbf{X}^f) - \underline{\mathbf{m}}) \rangle + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}} \rangle = \langle \underline{\mathbf{t}}, \underline{\Delta}_{s,u}(\mathbf{X}^f) \rangle.$$

Thus the condition (1.1) holds true and  $\mathbf{X}^f$  is a harness. Similarly, one can verify that (1.1) holds when  $\varphi$  is a decreasing function.

We use (3.14) to compute the mean of  $\mathbf{X}^f$

$$\mathbb{E}(\mathbf{X}^f(t)) = (ct + d)\langle \underline{\mathbf{t}}', \underline{\mu} \rangle + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}} \rangle = \langle A\underline{\mathbf{t}}, \underline{\mu} \rangle + \langle \underline{\mathbf{t}}, \underline{\mathbf{m}} \rangle = \langle \underline{\mathbf{t}}, \underline{\mathbf{m}} + A^T \underline{\mu} \rangle,$$

and (3.5) follows.

To find the covariance we again use (3.14) and the fact that  $\text{Cov}(X_{s'}, X_{t'})$  is either  $\langle \underline{\mathbf{s}}', \Sigma \underline{\mathbf{t}}' \rangle$  or  $\langle \underline{\mathbf{t}}', \Sigma \underline{\mathbf{s}}' \rangle = \langle \underline{\mathbf{s}}', \Sigma^T \underline{\mathbf{t}}' \rangle$  depending whether  $s' < t'$  (case  $\det(A) > 0$ ) or  $s' > t'$  (case  $\det(A) < 0$ ). For example, if  $\det(A) > 0$  then

$$\begin{aligned} \text{Cov}(\mathbf{X}^f(s), \mathbf{X}^f(t)) &= (cs + d)(ct + d)\text{Cov}(X_{s'}, X_{t'}) \\ &= (cs + d)(ct + d)\langle \underline{\mathbf{s}}', \Sigma \underline{\mathbf{t}}' \rangle = \langle A\underline{\mathbf{s}}, \Sigma A\underline{\mathbf{t}} \rangle = \langle \underline{\mathbf{s}}, A^T \Sigma A \underline{\mathbf{t}} \rangle, \end{aligned}$$

and thus (3.6) follows.

Since  $\phi$  is monotone on  $(T_0, T_1)$ ,

$$(3.16) \quad \text{Var}(\tilde{X}_t | \mathcal{F}_{s,u}^f) = \begin{cases} (ct + d)^2 \text{Var}(X_{t'} | \mathcal{F}_{s',u'}^f) & \text{if } \det(A) > 0, \\ (ct + d)^2 \text{Var}(X_{t'} | \mathcal{F}_{u',s'}^f) & \text{if } \det(A) < 0. \end{cases}$$

Consider the case  $\det(A) < 0$  so that  $u' < t' < s'$ . Since  $\underline{\Delta}_{a,b} = \underline{\Delta}_{b,a}$ , by (3.16) and Lemma 3.3, the conditional variance is

$$(3.17) \quad \begin{aligned} \text{Var}(\tilde{X}_t | \mathcal{F}_{s,u}^f) &= (ct + d)^2 F_{t',u',s'} \left( \chi + \langle A^{-1} \underline{\theta}_X, \underline{\Delta}_{s,u}(\mathbf{X}^f) - \underline{\mathbf{m}} \rangle \right. \\ &\quad \left. + \langle \underline{\Delta}_{s,u}(\mathbf{X}^f) - \underline{\mathbf{m}}, A^{-1} \Gamma_X (A^{-1})^T (\underline{\Delta}_{s,u}(\mathbf{X}^f) - \underline{\mathbf{m}}) \rangle \right). \end{aligned}$$

So (3.17) rewrites as

$$\begin{aligned} \text{Var}(\tilde{X}_t | \mathcal{F}_{s,u}^f) &= F_{t',u',s'} \left( \chi - \langle A^{-1} \underline{\theta}_X, \underline{\mathbf{m}} \rangle + \langle \underline{\Delta}_{s,u}(\tilde{\mathbf{X}}), A^{-1} \underline{\theta}_X \rangle \right. \\ &\quad \left. + \langle \underline{\Delta}_{s,u}(\tilde{\mathbf{X}}) - \underline{\mathbf{m}}, A^{-1} \Gamma (A^{-1})^T (\underline{\Delta}_{s,u}(\tilde{\mathbf{X}}) - \underline{\mathbf{m}}) \rangle \right) \\ &= F_{t',u',s'} \left( \chi - \langle A^{-1} \underline{\theta}_X, \underline{\mathbf{m}} \rangle + \langle \underline{\mathbf{m}}, \tilde{\Gamma}^T \underline{\mathbf{m}} \rangle \right. \\ &\quad \left. + \langle \underline{\Delta}_{s,u}(\tilde{\mathbf{X}}), A^{-1} \underline{\theta} - (\tilde{\Gamma} + \tilde{\Gamma}^T) \underline{\mathbf{m}} \rangle + \langle \underline{\Delta}_{s,u}(\tilde{\mathbf{X}}), \tilde{\Gamma} \underline{\Delta}_{s,u}(\tilde{\mathbf{X}}) \rangle \right) \\ &= \frac{\langle \underline{\mathbf{s}}, J \underline{\mathbf{t}} \rangle \langle \underline{\mathbf{t}}, J \underline{\mathbf{u}} \rangle}{\langle \underline{\mathbf{s}}, J^T \tilde{\Gamma} J \underline{\mathbf{u}} \rangle} \frac{1}{G} \left( \tilde{\chi} + \langle \underline{\Delta}_{s,u}(\tilde{\mathbf{X}}), A^{-1} \underline{\theta} - (\tilde{\Gamma} + \tilde{\Gamma}^T) \underline{\mathbf{m}} \rangle \right. \\ &\quad \left. + \langle \underline{\Delta}_{s,u}(\tilde{\mathbf{X}}), \tilde{\Gamma}^T \underline{\Delta}_{s,u}(\tilde{\mathbf{X}}) \rangle \right). \end{aligned}$$

Since the last term is invariant under transposition, we get (3.8) and (3.7). The case  $\det(A) > 0$  is handled similarly and the proof is omitted.  $\square$

The proof of Proposition 3.2 is based on the formula for the covariance matrix of vector  $\underline{\Delta}_{s,u}$ .

**Lemma 3.4.**

$$(3.18) \quad \text{Cov} \underline{\Delta}_{s,u} = \frac{c_1 - c_2}{u - s} J \underline{\mathbf{u}} \underline{\mathbf{s}}^T J^T + \Sigma^T.$$

*Proof.* From (3.13) we get

$$\begin{aligned} \text{Cov } \underline{\Delta}_{s,u} &= \mathbb{E} \left( \underline{\Delta}_{s,u} \underline{\Delta}_{s,u}^T \right) - \mathbb{E} \left( \underline{\Delta}_{s,u} \right) \mathbb{E} \left( \underline{\Delta}_{s,u}^T \right) \\ &= \frac{J \underline{s} \underline{u}^T \Sigma \underline{u} \underline{s}^T J^T - J \underline{u} \underline{s}^T \Sigma \underline{u} \underline{s}^T J^T + J \underline{u} \underline{s}^T \Sigma \underline{s} \underline{u}^T J^T - J \underline{s} \underline{s}^T \Sigma \underline{u} \underline{u}^T J^T}{(u-s)^2}. \end{aligned}$$

Note that since  $\underline{s}^T \Sigma \underline{u} = \underline{u}^T \Sigma^T \underline{s}$  and  $\underline{s} \underline{u}^T - \underline{u} \underline{s}^T = (u-s) J^T$ , the numerator can be written as

$$\begin{aligned} J(\underline{s} \underline{u}^T - \underline{u} \underline{s}^T) \Sigma \underline{u} \underline{s}^T J^T - J(\underline{s} \underline{u}^T - \underline{u} \underline{s}^T) \Sigma^T \underline{s} \underline{u}^T J^T &= (u-s)(\Sigma \underline{u} \underline{s}^T - \Sigma^T \underline{s} \underline{u}^T) J^T \\ \text{(recall that } J^T = -J \text{ and } J J^T = I). \text{ Further we write the above expression as} \\ (u-s)[(\Sigma - \Sigma^T) \underline{u} \underline{s}^T + \Sigma^T(\underline{u} \underline{s}^T - \underline{s} \underline{u}^T)] J^T &= (u-s)(c_1 - c_2) J \underline{u} \underline{s}^T J^T + (u-s)^2 \Sigma^T J J^T \\ \text{and thus (3.18) follows.} \quad \square \end{aligned}$$

*Proof of Proposition 3.2.* We first remark that formulas (3.7) and (3.6) imply that

$$\text{tr}(\Gamma \Sigma^T) = \text{tr}(\tilde{\Gamma} \tilde{\Sigma}).$$

Next, we note that

$$\chi + \langle \underline{\theta}, \underline{\mu} \rangle + \langle \underline{\mu}, \Gamma \underline{\mu} \rangle = \tilde{\chi} + \langle \tilde{\underline{\theta}}, \tilde{\underline{\mu}} \rangle + \langle \tilde{\underline{\mu}}, \tilde{\Gamma} \tilde{\underline{\mu}} \rangle.$$

(This follows from a longer calculation based on the formulas from Proposition 3.9.) Therefore, transformation formulas preserve (3.11).

Next, we show that (3.11) implies (2.9). This will be accomplished by computing the averages of both sides of (3.4).

For any harness with covariance (2.4),

$$(3.19) \quad \mathbb{E}(\text{Var}(X_t | \mathcal{F}_{s,u})) = \frac{(t-s)(u-t)}{u-s} (c_1 - c_2).$$

To prove (3.19), we use (3.18). From (2.1) we get

$$\begin{aligned} \mathbb{E} \text{Var}(X_t | \mathcal{F}_{s,u}) &= \text{Var } X_t - \text{Var}(\mathbb{E}(X_t | \mathcal{F}_{s,u})) = \text{Var } X_t - \underline{t}^T \text{Cov}(\underline{\Delta}_{s,u}) \underline{t} \\ &= \underline{t}^T \Sigma \underline{t} - \frac{c_1 - c_2}{u-s} \underline{t}^T J \underline{u} \underline{s}^T J^T \underline{t} - \underline{t}^T \Sigma^T \underline{t} = \frac{c_1 - c_2}{u-s} (\underline{s}^T J \underline{t})(\underline{t}^T J \underline{u}) = \frac{c_1 - c_2}{u-s} (t-s)(u-s). \end{aligned}$$

On the other hand, with  $K(\underline{\Delta}_{s,u}) = \chi + \langle \underline{\theta}, \underline{\Delta}_{s,u} \rangle + \langle \underline{\Delta}_{s,u}, \Gamma \underline{\Delta}_{s,u} \rangle$ , we have

$$(3.20) \quad \mathbb{E}(K(\underline{\Delta}_{s,u})) = \text{tr}(\Gamma \Sigma^T) + K(\underline{\mu}) + (c_1 - c_2) \frac{\langle \underline{s}, J^T \Gamma J \underline{u} \rangle}{u-s}.$$

To prove (3.20) we note that  $\mathbb{E} \underline{\Delta}_{s,u} = \underline{\mu}$ , so

$$\begin{aligned} (3.21) \quad \mathbb{E} K(\underline{\Delta}_{s,u}) &= \chi + \mathbb{E} \theta^T \underline{\Delta}_{s,u} + \mathbb{E} \underline{\Delta}_{s,u}^T \Gamma \underline{\Delta}_{s,u} \\ &= \chi + \theta^T \underline{\mu} + \underline{\mu}^T \Gamma \underline{\mu} + \text{tr}(\Gamma \text{Cov } \underline{\Delta}_{s,u}) = K(\underline{\mu}) + \text{tr}(\Gamma \text{Cov } \underline{\Delta}_{s,u}). \end{aligned}$$

From (3.18) we get

$$(3.22) \quad \text{tr}(\Gamma \text{Cov } \underline{\Delta}_{s,u}) = \frac{c_1 - c_2}{u-s} \text{tr}(\Gamma J \underline{u} \underline{s}^T J^T) + \text{tr}(\Gamma \Sigma^T).$$

Since

$$\text{tr}(\Gamma J \underline{u} \underline{s}^T J^T) = \text{tr}(\underline{s}^T J^T \Gamma J \underline{u}) = \langle \underline{s}, J^T \Gamma J \underline{u} \rangle,$$

(3.20) follows from (3.21) and (3.22).

Since  $\mathbb{E}(\text{Var}(X_t|\mathcal{F}_{s,u})) = F_{t,s,u}\mathbb{E}(K(\underline{\Delta}_{s,u}))$ , in the non-degenerate case  $c_1 > c_2$ , and in particular in the non-degenerate product (3.10) with  $\det \Theta > 0$ , formula (3.19) implies that  $\mathbb{E}(K(\underline{\Delta}_{s,u})) \neq 0$  so  $F_{t,s,u}$  is uniquely determined.

Moreover, while the quadratic part of  $K$  does not determine the entries of  $\Gamma$  uniquely, it is natural to choose a unique, perhaps non-symmetric, matrix  $\Gamma$  such that  $\text{tr}(\Gamma\Sigma^T) + K(\underline{\mu}) = 0$ , i.e. (3.11) holds. Then  $F_{t,s,u}$  is given by formula (2.9) which is just a matrix form of (1.3).  $\square$

*Remark 3.1.* It is interesting to remark that in the non-degenerate product case (3.10) with  $\det \Theta > 0$ , formula (3.19) takes the form

$$\mathbb{E}(\text{Var}(X_t|\mathcal{F}_{s,u})) = \frac{(t-s)(u-t)}{u-s} \det \Theta > 0.$$

*Remark 3.2.* If  $c_3 \geq 0$ ,  $c_1 > c_2$ , and  $c_0c_3 > c_2^2$  then the right hand side of (2.4) indeed defines a positive definite function on  $T = (0, \infty)$ . To verify this, let  $a(s, t) = \langle \underline{s}, \Sigma \underline{t} \rangle$  for  $s < t$  and  $c(s, t) = a(s \wedge t, s \vee t)$ . For  $s_0 = 0 < s_1 < \dots < s_n$ , we compute

$$\begin{aligned} \det[c(s_i, s_j)]_{0 \leq i, j \leq n} \\ = (c_1 - c_2)^n (s_n - s_{n-1}) \dots (s_2 - s_1) s_1 (c_3(c_1 - c_2) + (c_0c_3 - c_2^2)s_n). \end{aligned}$$

Then  $[c(s_i, s_j)]_{1 \leq i, j \leq n}$  is positive definite as a sub-matrix of the above.

From the proof of Proposition 3.2 we can also read out that processes with  $c_1 = c_2$  are degenerate in the sense that  $X_t$  is a linear combination of  $X_s, X_u$ .

### 3.2. Proofs of Theorems 1.1 and 1.2'.

*Proof of Theorem 1.1.* Let  $A = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & d \end{bmatrix}$  so that its inverse is  $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . We apply Proposition 3.1 with  $f([x, y]) = ([x, y] - [\alpha, \beta])B$  to

$$\underline{\mu} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}, \quad \Sigma = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \tau & \rho \\ \rho & \sigma \end{bmatrix}, \quad \underline{\theta} = \begin{bmatrix} \theta \\ \eta \end{bmatrix}.$$

From the transformation formulas we get  $\tilde{\mu} = 0$ ,  $\tilde{\Sigma} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , and  $\tilde{\chi} = \chi + \alpha\eta + \theta\beta + \sigma\alpha^2 + \tau\beta^2 + 2\rho\alpha\beta > 0$  by (1.7). We also get

$$\tilde{\underline{\theta}} = \begin{bmatrix} b(\eta + \beta\rho + 2\alpha\sigma) + a(\theta + \alpha\rho + 2\beta\tau) \\ d(\eta + \beta\rho + 2\alpha\sigma) + c(\theta + \alpha\rho + 2\beta\tau) \end{bmatrix}$$

and

$$\tilde{\Gamma} = \begin{bmatrix} \tau a^2 + b\rho a + b^2\sigma & \frac{1}{2}(bc\rho + ad\rho + 2bd\sigma + 2ac\tau) \\ \frac{1}{2}(bc\rho + ad\rho + 2bd\sigma + 2ac\tau) & \tau c^2 + d\rho c + d^2\sigma \end{bmatrix}.$$

The quadratic polynomial  $K$  remains unchanged if we replace  $\tilde{\Gamma}$  by

$$\Gamma' = \begin{bmatrix} \tau a^2 + b\rho a + b^2\sigma & -\tilde{\chi} \\ \tilde{\chi} + bc\rho + ad\rho + 2bd\sigma + 2ac\tau & \tau c^2 + d\rho c + d^2\sigma \end{bmatrix}$$

Rewriting the quadratic polynomial as

$$K(\underline{x}) = \tilde{\chi} + \langle \tilde{\underline{\theta}}, \underline{x} \rangle + \langle \underline{x}, \Gamma' \underline{x} \rangle = \tilde{\chi} \left( 1 + \langle \frac{1}{\tilde{\chi}} \tilde{\underline{\theta}}, \underline{x} \rangle + \langle \underline{x}, \frac{1}{\tilde{\chi}} \Gamma' \underline{x} \rangle \right),$$

we get the parameters as claimed.  $\square$

*Proof of Theorem 1.2'.* To prove that  $V(1 + R\sigma) + \tau - R\gamma > 0$  we note that  $V \mapsto V(1 + R\sigma) + \tau - R\gamma$  is a continuous function on  $\mathbb{R}$  which by Proposition 3.2 cannot cross zero on  $(R, \infty)$ .

The remainder of the proof consists of computing the parameters of  $\mathbf{X}$  so that we can apply Theorem 1.1. First, from (2.1) we see that the mean of  $\mathbf{X}$  is given by  $\underline{\mu} = \begin{bmatrix} \Delta_{RV} \\ \tilde{\Delta}_{RV} \end{bmatrix}$ . Next, from (2.10) we see that  $\text{Cov}(X_s, X_t) = M^2(V - t)(s - R)$  with

$$(3.23) \quad M = \frac{\sqrt{K(\Delta_{RV}, \tilde{\Delta}_{RV})}}{\sqrt{V(1 + R\sigma) + \tau - R\gamma}} > 0.$$

Finally, we observe that the conditional variance of  $\mathbf{X}$  is given by the same formula as the conditional variance for  $\mathbf{Z}$ . Therefore we can apply Theorem 1.1 with

$$\alpha = \tilde{\Delta}_{RV}, \beta = \Delta_{RV}, \chi = 1, \\ a = M\sqrt{V}, b = -RM\sqrt{V}, c = -M/\sqrt{V}, d = M\sqrt{V}.$$

(Other choices are also possible and lead to "equivalent" quadratic harnesses as in (1.14).) Under the above choice of  $a, b, c, d$ , formula (1.8) gives (1.17). Since  $\tilde{\chi} = K(\Delta_{RV}, \tilde{\Delta}_{RV}) > 0$ , assumption (1.7) holds, and the parameters of the resulting quadratic harness are as claimed.  $\square$

#### 4. CONDITIONING OF QUADRATIC HARNESSES

In this section we show how to apply previous results to analyze which quadratic harnesses can be constructed from already known quadratic harnesses either by conditioning. Our basic building blocks will be the  $q$ -Meixner processes: these are quadratic harnesses with  $\eta = \sigma = 0$  and  $\gamma = q \in [-1, 1]$ , see [5]. In particular, the five classical Lévy processes mentioned in the introduction, sometimes called Meixner processes, correspond to  $\gamma = 1$ .

Our main interest in such constructions stems from the fact that the constructions from the Askey-Wilson laws [1] yield only  $\gamma \in (-1, 1 - 2\sqrt{\sigma\tau})$ ; in particular "classical" quadratic harnesses with the boundary value of  $\gamma = 1 - 2\sqrt{\sigma\tau}$  that appear in [3, Proposition 4.4] need additional work and have been constructed only for particular choices of the parameters. Example 4.1 and Example 4.2 present two new quadratic harnesses with  $\gamma = 1 - 2\sqrt{\sigma\tau}$ , and in Remark 4.1 we mention two more cases.

**4.1. Conditioning of Meixner processes.** We can use Theorem 1.2' to recognize which quadratic harnesses could arise by conditioning from quadratic harnesses that are Lévy processes.

**Proposition 4.1.** *Fix  $\tau \geq 0$  and  $\theta \in \mathbb{R}$ . Suppose  $\mathbf{Z} \in QH(0, \theta; 0, \tau; 1)$  i.e.,  $\mathbf{Z}$  is a Meixner process. If  $\mathbf{X}$  is a conditional bridge of  $\mathbf{Z}$  then*

$$(4.1) \quad Y_t = \frac{\sqrt{V - R + \tau}}{(V - R)\sqrt{1 + \theta\Delta_{RV} + \tau\Delta_{RV}^2}} ((1 + t)X_{(Vt+R)/(1+t)} - tz_V - z_R)$$

*is in  $QH(\eta_Y, \theta_Y; \sigma_Y, \tau_Y; \gamma_Y)$  with parameters*

$$(4.2) \quad \gamma_Y = \frac{V - R - \tau}{V - R + \tau}, \tau_Y = \sigma_Y = \frac{\tau}{V - R + \tau},$$

$$(4.3) \quad \theta_Y = -\eta_Y = \frac{\theta + 2\tau\Delta_{RV}}{\sqrt{V-R+\tau}\sqrt{1+\theta\Delta_{RV}+\tau\Delta_{RV}^2}}.$$

*Proof.* From Theorem 1.2' we read out the transformation that leads to parameters

$$\begin{aligned} \gamma' &= \frac{V-R-\tau_Z}{V-R+\tau}, \quad \tau' = \frac{\tau V}{V-R+\tau}, \quad \sigma' = \frac{\tau}{V(V-R+\tau)}, \\ \theta' &= \sqrt{V} \frac{\theta + 2\tau\Delta_{RV}}{\sqrt{V-R+\tau}\sqrt{1+\theta\Delta_{RV}+\tau\Delta_{RV}^2}}, \\ \eta' &= -\frac{\theta + 2\tau\Delta_{RV}}{\sqrt{V}\sqrt{V-R+\tau}\sqrt{1+\theta\Delta_{RV}+\tau\Delta_{RV}^2}}. \end{aligned}$$

Further transformation (1.14) leads to (4.1) and to parameters as claimed.  $\square$

**Example 4.1** (Dirichlet process). For any  $\sigma_0, \tau_0 > 0$  with  $\sigma_0\tau_0 < 1$ , there exists a quadratic harness  $\mathbf{Y}$  (namely, a Dirichlet process) which has parameters  $\sigma_Y = \sigma_0$ ,  $\tau_Y = \tau_0$ ,  $\theta_Y = 2\sqrt{\tau_0}$ ,  $\eta_Y = -2\sqrt{\sigma_0}$ , and  $\gamma_Y = 1 - 2\sqrt{\sigma_0\tau_0}$ .

Indeed, consider the case of conditional harnesses obtained from a gamma process  $(G_t)_{t>0}$ . Gamma process  $(G_t)$  is a non-negative two-parameter Lévy process with parameters  $\alpha, \beta > 0$ . The density of  $G_t$  given by

$$\beta^{\alpha t} x^{\alpha t-1} e^{-\beta x} / \Gamma(\alpha) \mathbf{1}_{(0,\infty)}(x).$$

As a Lévy process,  $(G_t)$  is a harness with mean  $\mathbb{E}(G_t) = t\alpha/\beta$  and variance  $\text{Var}(G_t) = t\alpha/\beta^2$ . It is also known (see (4.6) below) that

$$(4.4) \quad \text{Var}(G_t | \mathcal{F}_{s,u}) = \frac{(t-s)(u-t)}{(u-s)^2((u-s)/\alpha + 1)} (G_u - G_s)^2.$$

Then

$$Z_t = \frac{\beta}{\alpha} G_{\alpha t} - \alpha t$$

is a quadratic harness in  $QH(0, 2/\alpha; 0, 1/\alpha^2; 1)$  which by further transformation (1.14) can be transformed into an element of  $QH(0, 2; 0, 1; 1)$ .

Instead of considering a conditional process of  $(G_t)$  we therefore consider a conditional process of  $\mathbf{Z} \in QH(0, 2; 0, 1; 1)$  and apply Proposition 4.1. Choose  $V-R = \sqrt{\sigma_0\tau_0} - \sigma_0\tau_0$  so that  $1/(1+V-R) = \sqrt{\sigma_0\tau_0}$ . From (4.2) we have

$$\gamma_Y = 1 - \frac{2}{V-R+1} = 1 - 2\sqrt{\sigma_0\tau_0},$$

Transformation (1.14) with  $a^2 = \sqrt{\sigma_0/\tau_0}$  gives

$$\tau_Y = \sqrt{\sigma_0\tau_0}/a^2 = \tau_0, \quad \sigma_Y = a^2\sqrt{\sigma_0\tau_0} = \sigma_0.$$

By (4.3) with any  $\Delta := \Delta_{RV} \geq 0$

$$a\theta_Y = \frac{2(1+\Delta)}{\sqrt{V-R+1}\sqrt{1+2\Delta+\Delta^2}} = \frac{2}{\sqrt{V-R+1}} = 2\sqrt{\sigma_0\tau_0},$$

so  $\theta_Y = 2\sqrt{\tau_0}$ . Similarly,  $\eta_Y/a = -2\sqrt{\sigma_0\tau_0}$ , so  $\eta_Y = -2\sqrt{\sigma_0}$ .

Since conditional processes of the gamma process are Dirichlet processes, the same conclusion can be obtained directly by a fairly natural reparametrization (4.7) without invoking explicitly any of the transformations. Let  $a_1, \dots, a_n, a_{n+1}$  be



positive numbers. A Dirichlet distribution  $D_n(a_1, \dots, a_n, a_{n+1})$  is defined through its density

$$f(x_1, \dots, x_n) = \frac{\Gamma(a_1 + \dots + a_{n+1})}{\prod_{i=1}^{n+1} \Gamma(a_i)} \prod_{i=1}^n x_i^{a_i-1} \left(1 - \sum_{i=1}^n x_i\right)^{a_{n+1}} \mathbf{1}_{U_n}(x_1, \dots, x_n),$$

where  $U_n = \{(x_1, \dots, x_n) \in (0, \infty)^n : \sum_{i=1}^n x_i < 1\}$ . A stochastic process  $\mathbf{X} = (X_t)_{t \in [0, V]}$  is called a Dirichlet process if there exists a finite nonzero measure  $\mu$  on  $[0, V]$  such that for any  $n$  and any  $0 \leq t_1 < \dots < t_n \leq V$  the distribution of the vector of increments  $(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$  is Dirichlet

$$D_n(\mu([0, t_1]), \mu([t_1, t_2]), \dots, \mu([t_{n-1}, t_n]), \mu([t_n, V])).$$

This is one of the basic objects of non-parametric Bayesian statistics - see [10, 12]. Let  $\mu = c\lambda$ , where  $\lambda$  is a Lebesgue measure on  $[0, V]$  and  $c = 1/\alpha > 0$  is a number. Recall that the beta distribution,  $B_I(a, b)$ , is defined by the density

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1} \mathbf{1}_{(0,1)}(x),$$

and if  $X \sim B_I(a, b)$  then

$$(4.5) \quad \mathbb{E}(X) = \frac{a}{a+b}, \quad \text{and} \quad \mathbb{V}\text{ar}(X) = \frac{ab}{(a+b)^2(a+b+1)}.$$

Since  $X_t$  has the beta distribution  $B_I(ct, c(V-t))$  the formulas (4.5) give

$$\mathbb{E}(X_t) = \frac{t}{V} \quad \text{and} \quad \mathbb{C}\text{ov}(X_s, X_t) = \frac{s(V-t)}{V^2(cT+1)}.$$

Note that to compute  $\mathbb{E}(X_s X_t)$  it is convenient to use the classical fact, that  $X_s/X_t$  and  $X_t$  are independent and  $X_s/X_t$  is a beta  $B_I(cs, c(t-s))$  random variable. Note also that  $\mathbf{X}$  is Markov process with transition distribution defined by the fact that  $(X_t - X_s)/(1 - X_s)$  and  $X_s$  are independent, and  $(X_t - X_s)/(1 - X_s)$  is beta  $B_I(c(t-s), c(V-t))$ . It is also known that

$$\frac{X_t - X_s}{X_u - X_s} \sim B_I(c(t-s), c(u-t))$$

and  $(X_t - X_s)/(X_u - X_s)$  and  $(X_s, X_u)$  are independent. Therefore, from (4.5) we get

$$\mathbb{E}(X_t | \mathcal{F}_{s,u}) = X_s + (X_u - X_s) \frac{t-s}{u-s}$$

and thus  $\mathbf{X}$  is a harness. The second formula in (4.5) gives

$$(4.6) \quad \mathbb{V}\text{ar}(X_t | \mathcal{F}_{s,u}) = (X_u - X_s)^2 \frac{(t-s)(u-t)}{(u-s)^2(c(u-s)+1)}.$$

Define now

$$(4.7) \quad Y_t = \sqrt{c + \frac{1}{V}} \left( (V+t)X_{\frac{tV}{V+t}} - t \right), \quad t \in [0, \infty).$$

It is elementary to check that  $(Y_t)_{t \in [0, \infty)}$  is a quadratic harness and the parameters are as follows

$$\theta_Y = \frac{2\sqrt{V}}{\sqrt{1+cV}}, \quad \eta_Y = \frac{-2}{\sqrt{V}\sqrt{1+cV}},$$

$$\tau_Y = \frac{V}{1+cV}, \quad \sigma_Y = \frac{1}{V(1+cV)}, \quad \gamma_Y = 1 - \frac{2}{(1+cV)^2}.$$

(Note that this agrees with the answers deduced from Proposition 4.1 which implies that  $\theta_Y^2 = 4\tau_Y$ ,  $\eta_Y^2 = 4\sigma_Y$  and  $\gamma_Y = 1 - 2\sqrt{\sigma_Y\tau_Y}$ .)

On the other hand, it can be easily seen that the process  $\mathbf{X}$  is a bridge on the gamma process  $(G_t)_{t \in [0, \infty)}$  governed by the gamma distribution with the shape parameter  $1/c$  and the scale equal 1. More precisely, in distribution process  $\mathbf{X}$  is identical to the gamma bridge  $(G_t/G_V)_{t \in [0, V]} | G_V \stackrel{d}{=} (G_t/G_V)_{t \in [0, V]}$ , see [13, Definition 2], see also [11].

**Example 4.2** (Binomial process). For any real  $\eta_0, \theta_0$ , such that  $\eta_0\theta_0 = -1/N < 0$  for some  $N \in \mathbb{N}$  there exist a quadratic harness  $\mathbf{Y}$  (namely, the Binomial process described here) which has parameters  $\sigma_Y = \tau_Y = 0$ ,  $\theta_Y = \theta_0$ ,  $\eta_Y = \eta_0$ , and  $\gamma_Y = 1$ .

Indeed, consider conditional harnesses obtained as bridges from a Poisson process. Poisson process  $N_t$  with parameter  $\lambda > 0$  is a harness with mean  $\mathbb{E}(N_t) = \lambda t$  variance  $\text{Var}(N_t) = \lambda t$ , and with conditional variance with respect to natural  $\sigma$ -fields given by

$$\mathbb{V}\text{ar}(N_t | \mathcal{F}_{s,u}) = \frac{(u-t)(t-s)}{(u-s)^2} (N_u - N_s).$$

Then

$$Z_t = N_{t/\lambda} - t$$

is in  $QH(0, 1; 0, 0; 1)$ .

Instead of considering a conditional process of  $(N_t)_{t>0}$  we therefore consider a conditional process of  $\mathbf{Z}$  and apply Proposition 4.1. From (4.2) we see that  $\gamma_X = 1$ ,  $\sigma_X = \tau_X = 0$ . Indeed, to simplify the notation, consider  $R = 0$ . Since  $\Delta_{0V} = (N - V)/V$  for some  $N \in \mathbb{N}$ , from (4.3) we see that  $\theta_Y = -\eta_Y = 1/\sqrt{N}$ . So  $\eta_Y\theta_Y = -1/N$  for some integer  $N \in \mathbb{N}$ .

The conditional processes of the Poisson process are the Binomial processes. So the same conclusion can be obtained more directly without invoking explicitly any of the transformations. (Compare [5, Proposition 4.4].) Let  $b(n, p)$  denote the binomial distribution with sample size  $n$  and probability of success  $p$ . For fixed  $N \in \mathbb{N}$ , define a Markov process  $\mathbf{X} = (X_t)_{t \in [0, V]}$  by the following (consistent) family of marginal and conditional distributions:

$$X_t \sim b\left(N, \frac{t}{V}\right) \quad \text{and} \quad X_t - X_s | Y_s \sim b\left(N - Y_s, \frac{t-s}{V-s}\right), \quad 0 \leq s \leq t \leq V.$$

Then the process  $\mathbf{X}$  is called a binomial process with parameter  $N$ . It is elementary to see that the conditional distribution  $X_t | \mathcal{F}_{s,u} \sim b\left(Y_u - Y_s, \frac{t-s}{u-s}\right)$ . Therefore  $\mathbf{X}$  is a harness i.e. (1.1) holds, and for any  $s, t, u \in [0, V]$ ,  $s < t < u$

$$\mathbb{V}\text{ar}(X_t | \mathcal{F}_{s,u}) = \frac{(u-t)(t-s)}{(u-s)^2} (X_u - X_s).$$

An easy computation (or an application of Theorem 1.1) shows that if

$$Y_t = \frac{(V+t)X_{\frac{tV}{V+t}} - tN}{\sqrt{VN}}, \quad t \in [0, \infty),$$

then the process  $(Y_t)_{t \geq 0}$  is a quadratic harness and the parameters are  $\theta_Y = \sqrt{V/N}$ ,  $\eta_Y = -1/\sqrt{VN}$ ,  $\tau_Y = \sigma_Y = 0$  and  $\gamma_Y = 1$ .

On the other hand, it is immediate that  $\mathbf{X}$  is a bridge obtained by conditioning a Poisson process  $(N_t)_{t \geq 0}$ , that is  $\mathbf{X} \stackrel{d}{=} (N_t)_{t \in [0, V]} | N_V = N$ .

It is interesting to see which properties of a quadratic harness are preserved by conditioning. The only universal invariant that we found is related to parameter  $q$  of the Askey-Wilson law from the construction in [1].

**Proposition 4.2.** *Suppose that  $\mathbf{Y} \in QH(\eta_Y, \theta_Y; \sigma_Y, \tau_Y; \gamma_Y)$  is a quadratic harness arising from a conditional bridge  $\mathbf{X}$  in  $\mathbf{Z} \in QH(\eta_Z, \theta_Z; \sigma_Z, \tau_Z; \gamma_Z)$ .*

(i)  $\gamma_Z = -1$  if and only if  $\gamma_Y = -1$ .

(ii) If  $\gamma_Z > -1$ , then

$$(4.8) \quad \frac{(1 - \gamma_Y)^2 - 4\sigma_Y\tau_Y}{(1 + \gamma_Y)^2} = \frac{(1 - \gamma_Z)^2 - 4\sigma_Z\tau_Z}{(1 + \gamma_Z)^2}.$$

(iii) If  $\mathbf{Z}$  is a  $q$ -Meixner process, i.e.  $\eta_Z = \sigma_Z = 0$ ,  $\gamma = q \in [-1, 1]$  and the conditioning is with  $R = 0$  and  $z_R = 0$ , then

$$(4.9) \quad \text{sign}(\theta_Y^2 - 4\tau_Y) = \text{sign}(\theta_Z^2 - 4\tau_Z).$$

*Proof.* The first statement follows from (1.23). Formula (4.8) follows by a direct computation from (1.21–1.23). Formula (4.9) follows from the expressions in Remark 1.2, which gives  $\theta_Y^2 - 4\tau_Y = (\theta_Z^2 - 4\tau_Z)/(V + \tau_Z)$ .  $\square$

*Remark 4.1.* For a Meixner process  $\mathbf{Z}$ , the conclusion of Proposition 4.2(iii) can be strengthened to double equality

$$\text{sign}(\eta_Y^2 - 4\sigma_Y) = \text{sign}(\theta_Y^2 - 4\tau_Y) = \text{sign}(\theta_Z^2 - 4\tau_Z).$$

Indeed, from Proposition 4.1 we see that  $\eta_Y^2 - 4\sigma_Y = (\theta_Y^2 - 4\tau_Y)\sigma_Y/\tau_Y$ .

In particular, quadratic harness  $\mathbf{Y}$  that arises by conditioning from the negative binomial process  $\mathbf{Z}$  has parameters  $\eta_Y\sqrt{\tau_Y} + \theta_Y\sqrt{\sigma_Y} = 0$ ,  $\gamma_Y = 1 - 2\sqrt{\sigma_Y\tau_Y}$ , and  $\theta_Y^2 > 4\tau_Y$ . (In fact,  $\mathbf{Y}$  rises from a discrete “negative hypergeometric” processes.) Similarly, quadratic harness  $\mathbf{Y}$  that arises from a conditional bridge of the “hyperbolic Meixner” process  $\mathbf{Z}$  with  $\theta_Z^2 < 4\tau_Z$  has parameters  $\eta_Y\sqrt{\tau_Y} + \theta_Y\sqrt{\sigma_Y} = 0$ ,  $\gamma_Y = 1 - 2\sqrt{\sigma_Y\tau_Y}$ , and  $\theta_Y^2 < 4\tau_Y$ .

**4.2. Proof of Theorem 1.3.** From (1.14) it is enough to construct a quadratic harness with parameters  $\sigma_X = \tau_X \in (0, 1)$ ,  $\theta_X = -\eta_X$  and  $\gamma_X = 1 - 2\sigma_X$ . To do so, we determine the parameters of  $\mathbf{Z}$  from (4.2) and (4.3), noting that for a Meixner processes  $\Delta_{RV} = (Z_V - Z_R)/(V - R)$  can take any positive real value that we fix first.

## 5. GLUING CONSTRUCTION

This section is motivated by the construction of a classical bi-Poisson process from a pair of two conditionally independent Poisson processes [6] and by another recent construction of a quadratic harness from two conditionally independent copies of a negative binomial process [16]. These constructions essentially consist of choosing an appropriate deterministic moment of time  $V$  so that  $Z_V$ -conditional (and  $Y_V$ -conditionally independent) processes  $\mathbf{X}_+ := \{Z_t : t > V\} | Z_V$  and  $\mathbf{X}_- := \{Z_t : t < V\} | Z_V$  arise as space time transforms of the above mentioned Lévy processes.

We remark that in principle all quadratic harnesses arise from such a gluing construction. A fixed  $V > 0$  can be treated as the upper value  $V$  in Remark 1.2 resulting in the process  $\mathbf{X}_- = (Z_t)_{t \leq V}$  conditioned with respect to  $Z_V$ , and on the other hand, we can treat this value as the left-hand side value  $R$  in Remark 1.2 and consider the process  $\mathbf{X}_+ = (Z_t)_{t \geq R}$  conditioned with respect to  $Z_R$ . Then both processes are quadratic harnesses and by Markov property they are  $Z_V$ -conditionally independent. So for example, a Poisson process arises from gluing a binomial process with another Poisson process, or a Wiener process arises from gluing a Brownian bridge with another Wiener process.

The question of interest here is when the "components"  $\mathbf{X}_-$  and  $\mathbf{X}_+$  to be glued are in some sense "simpler" than the resulting process. Using Corollary 1.2' and Remark 1.2 we can recognize when the components of such a gluing construction come from a well understood class of  $q$ -Meixner processes. (Since we are proving necessity only, we do not analyze  $\mathbf{X}_+$ .)

**Proposition 5.1.** *Let  $\mathbf{Z} \in QH(\eta, \theta; \sigma, \tau, \gamma)$  be defined on  $(0, \infty)$ . Suppose that there is a deterministic moment of time  $V > 0$  such that Markov process  $\mathbf{X}_-$  obtained as  $\{Z_t : t < V\}$  conditioned with respect to  $Z_V$ , can be transformed via (1.27) into a  $q$ -Meixner process  $\mathbf{Y}$ , then one of the following cases must happen:*

- (i)  $\gamma = 1$ ,  $\sigma = \tau = 0$  and  $\eta = \theta = 0$ . (Then  $\mathbf{X}$  is the Wiener processes and the construction indeed works with any  $V > 0$ .)
- (ii)  $\gamma = 1$ ,  $\sigma = \tau = 0$  and  $\eta\theta > 0$ . (Then  $V = \theta/\eta$ ,  $\mathbf{X}$  is the Poisson processes with parameter  $\lambda$  which depends on  $Y_V$  and the construction indeed works, see [6].)
- (iii)  $\eta\sqrt{\tau} = \theta\sqrt{\sigma}$ , and  $\gamma = 1 + 2\sqrt{\sigma\tau} > 1$ . (Then  $V = \sqrt{\tau/\sigma}$  and  $\mathbf{X} \in QH(0, \theta_X; 0, \tau_X; 1)$  with the sign of  $\theta_X^2 - 4\tau_X$  determined by the sign of  $\theta^2 - 4\tau$  of process  $\mathbf{Y}$ .)

*Proof.* If  $\mathbf{Y}$  comes from gluing a  $q$ -Meixner process then the conditional bridge  $\mathbf{Y}_-$  corresponding to  $R = 0, V = V$  exists and can be transformed into a  $q$ -Meixner process  $\mathbf{X}$  with parameters given in Remark 1.2(i).

The only possibility for (1.31) to correspond to a  $q$ -Meixner process is when the parameters of  $\mathbf{Y}$  satisfy

$$(5.1) \quad \sigma V^2 + (1 - \gamma)V + \tau = 0.$$

Since  $\sigma, \tau \geq 0$  (see [3, Theorem 2.2]), the only solution with  $\gamma \leq 1$  is  $\gamma = 1$ ,  $\sigma = \tau = 0$ . Then from (1.29) we see that  $\mathbf{X}$  is indeed a Meixner process when we set  $V = \theta/\eta$  (which gives the bi-Poisson process) or when  $V > 0$  but  $\eta = \theta = 0$  (which gives the Wiener process).

Other solutions of (5.1) are possible only when  $(1 - \gamma)^2 \geq 4\sigma\tau$ . However, since  $\gamma \leq 1 + 2\sqrt{\sigma\tau}$  by [3, Theorem 2.2], this gives  $\gamma = 1 + 2\sqrt{\sigma\tau}$  and  $V = \sqrt{\tau/\sigma}$ . Then from (1.29), the coefficient at  $z_V$  vanishes so  $\mathbf{X}$  is indeed a Meixner process when  $\eta\sqrt{\tau} = \theta\sqrt{\sigma}$ .

For  $V = \sqrt{\tau/\sigma}$ ,  $\gamma = 1 + 2\sqrt{\sigma\tau}$ , formulas from Remark 1.2 (i) give

$$\frac{\theta_X^2}{4\tau_X} = \frac{\theta^2 + 4\theta\sqrt{\sigma\tau}z_V + 4\sigma\tau z_V^2}{4\tau + 4\theta\sqrt{\sigma\tau}z_V + 4\sigma\tau z_V^2},$$

so the sign of  $\theta^2 - 4\tau$  is preserved.  $\square$

We remark that [16] provided gluing construction of two conditionally-independent copies of negative binomial processes. The resulting quadratic harness on  $(0, \infty)$  reaches the “upper limit”  $\gamma = 1 + 2\sqrt{\sigma\tau}$  of the bound in [3, Theorem 2.2] and that for this process the product  $\sigma\tau$  can take arbitrarily values in  $(0, \infty)$ . We do not yet know whether processes in Proposition 5.1 indeed exist for all signs of  $\theta^2 - 4\tau$ .

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## APPENDIX A. AUXILIARY DETAILS

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